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$6D$ dyonic string with active hyperscalars

Der-Chyn Jong,[a] **Ali Kaya**[bc] **and Ergin Sezgin**[abc]

[a] *George P. and Cynthia W. Mitchell Institute for Fundamental Physics*
Texas A&M University
College Station, TX 77843-4242
[b] *Boğaziçi University, Department of Physics*
34342, Bebek, Istanbul, Turkey
[c] *Feza Gürsey Institute, 34684, Çengelköy*
Istanbul, Turkey
E-mail: djong@physics.tamu.edu, ali.kaya@boun.edu.tr, sezgin@physics.tamu.edu

Abstract: We derive the necessary and sufficient conditions for the existence of a Killing spinor in $N = (1, 0)$ gauge supergravity in six dimensions coupled to a single tensor multiplet, vector multiplets and hypermultiplets. These are shown to imply most of the field equations and the remaining ones are determined. In this framework, we find a novel 1/8 supersymmetric dyonic string solution with nonvanishing hypermultiplet scalars. The activated scalars parametrize a 4 dimensional submanifold of a quaternionic hyperbolic ball. We employ an identity map between this submanifold and the internal space transverse to the string worldsheet. The internal space forms a 4 dimensional analog of the Gell-Mann-Zwiebach tear-drop which is noncompact with finite volume. While the electric charge carried by the dyonic string is arbitrary, the magnetic charge is fixed in Planckian units, and hence necessarily non-vanishing. The source term needed to balance a delta function type singularity at the origin is determined. The solution is also shown to have 1/4 supersymmetric $AdS_3 \times S^3$ near horizon limit where the radii are proportional to the electric charge.

Keywords: Supergravity Models, p-branes, Sigma Models, Field Theories in Higher Dimensions.

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Contents

1. Introduction

Anomaly-free matter coupled supergravities in six dimensions naturally arise in $K3$ compactification of Type I and heterotic string theories [1]. Owing to the fact that $K3$ has no isometries, all of the resulting $6D$ models are ungauged in the sense that the R-symmetry group $\mathrm{Sp}(1)_R$, or its $\mathrm{U}(1)_R$ subgroup thereof, is only a global symmetry. The R-symmetry gauged general matter coupled models, on the other hand, have been constructed directly in six dimensions long ago [2, 3]. These theories harbor gravitational, gauge and mixed anomalies which can be encoded in an 8-form anomaly polynomial, and the Green-Schwarz anomaly cancelation mechanism requires its factorization. It turns out that the R-symmetry gauging reduces drastically the space of solutions to this requirement.

At present, the only known "naturally" anomaly-free gauged supergravities in $6D$ are:

- the $E_7 \times E_6 \times \mathrm{U}(1)_R$ invariant model in which the hyperfermions are in the $(912, 1, 1)$ representation of the gauge group [4],
- the $E_7 \times G_2 \times \mathrm{U}(1)_R$ invariant model with hyperfermions in the $(56, 14, 1)$ representation of the gauge group [5], and
- the $F_4 \times \mathrm{Sp}(9) \times \mathrm{U}(1)_R$ invariant model with hyperfermions in the $(52, 18, 1)$ representation of the gauge group [6].

The anomaly freedom of these models is highly nontrivial, and they are natural in the sense that they do not contain any gauge-singlet hyperfermions. If one considers a large factor of U(1) groups, and tune their U(1) charges in a rather ad-hoc way [6], or considers only products of SU(2) and U(1) factors with a large number of hyperfermions, and tune their U(1) charges again in an ad-hoc way, infinitely many possible anomaly-free combinations arise [7]. These models appear to be "unnatural" at this time.

In fact, none of the above mentioned models, natural or not, have any known string/M-theory origin so far, though progress has been made in embedding [8] a minimal sub-sector with $\mathrm{U}(1)_R$ symmetry and no hyperfermions [9] in string/M theory. An apparently inconclusive effort has also been made in [10] in which the $6D$ theory is considered to live on the boundary of a $7D$ theory, which, in turn is to be obtained from string/M-theory.

Finding the string/M-theory origin of the anomaly free models mentioned above is likely to uncover some interesting mechanisms for descending to lower dimensions starting from string/M-theory. Moreover, models of these type have been increasingly finding remarkable applications in cosmology and braneworld scenarios [11 − 16].

In this paper, we will not address the string/M-theory origin of the $6D$ theories at hand but rather investigate the general form of their supersymmetric solutions, and present, in particular, a dyonic string solution in which the hyperscalar fields have been activated. Our aims are:

- to lay out the framework for finding further solutions which, in turn, may lead to new solutions in other theories of interest that live in diverse dimensions,
- to establish the fact that (dyonic) string solution exists in a more general situation than so far that has been known, in the sense that new type of fields, to wit, hyper-scalars, have been activated, and
- to open new avenues in the compactification schemes in which the sigma model sector of supergravity theories are exploited.

These aims call for a modest summary of what has been done in these areas so far. To begin with, the general form of supersymmetric solutions in $6D$ have been studied in [17, 18], though in the absence of hypermultiplets. We will fill this gap here. We will extend the analysis for the existence of Killing spinors, determine the resulting integrability conditions and the necessary and sufficient equations for finding exact solutions, without having to directly solve all the field equations.

Second, various dyonic string solutions of $6D$ supergravities exist in the literature [19 – 22], though again, none of them employ the hypermatter. We will find some novel features here such as the necessity to switch on the magnetic charge of the dyonic string.

Third, concerning the use of (higher than one dimensional) sigma model sector of supergravity theories in finding exact solutions, in the case of ungauged supergravities the oldest result is due to Gell-Mann-Zwiebach [23] who found the half-supersymmetry breaking tear-drop solution of Type IIB supergravity, by exploiting its $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ sigma model sector. The tear-drop represents the two-dimensional internal space which is non-compact with finite volume. The sigma model sector of Type IIB supergravity has also been utilized in finding an instanton solution dual to a 7-brane [24]. Supersymmetric two dimensional tear-drop solutions in ungauged $D < 10$ supergravities are also known [23, 25, 14, 15, 26]. More recently, the general form of the supersymmetric solutions in ungauged $4D$ supergravities, including their coupling to hypermatter, have been provided in [27].

In the case gauged supergravities, a solution of the matter coupled $N = (1, 0)$ gauged supergravity in $6D$ called 'the superswirl' has been found in [28] where two hyperscalars are activated. One of these scalars is dilatonic and the other one is axionic. Supersymmetric domain-wall solutions of maximal gauged supergravities in diverse dimensions where only the dilatonic scalars of the sigma model are activated have appeared in [29]. Supersymmetric black string solutions of matter coupled $N = 2, D = 3$ gauged supergravity exists in which only a single dilaton is activated in the Kahler sigma model sector [30]. In such models, supersymmetric solutions with the additional axionic scalars activated, have also been found [31 – 33]. Finally, conditions for Killing spinors and general form of the supersymmetric solutions in matter coupled gauged supergravities in $N = 2, D = 5$ supergravities have also been investigated [34] but no specific solutions with multi-hyperscalars activated seem to have appeared.

To summarize, we see that there exist only few scattered results on the nontrivial use of gauged sigma models in supergravity theories in finding exact supersymmetric solutions. As stated earlier, one of our goals in this paper is to take a step towards a systematic approach to this problem. We shall come back to this point in the Conclusions.

Turning to the tear-drop solutions, a key feature in these backgrounds is the identity map by which the scalars of the sigma model manifold are identified with those of the internal part of the spacetime. The brief summary of literature above only dealt with solutions that have supersymmetry. The idea of identity map, on the other hand, was first proposed by Omero and Percacci [35] long ago in the context of bosonic sigma models coupled to gravity. This work was generalized later in [36]. Several more papers may well exist that deal with the solutions of sigma model coupled ordinary gravities, as opposed to supergravities, but we shall not attempt to survey them since our emphasis is on gauge supergravities with sigma model sectors in this paper.

After the description of the matter coupled $6D$ supergravity in the next section, the conditions for the existence of Killing spinors, and their integrability conditions will be presented in sections 3 and 4, respectively. The new dyonic string solution and its properties are then described in sections 5 and 6, respectively. The summary of our results that emphasizes the key points, and selected open problems are given in the Conclusions. Three appendices that contain our conventions and useful formulae are also presented.

2. The model

2.1 Field content and the quaternionic Kahler scalar manifold

The six-dimensional gauged supergravity model we shall study involves the combined $N = (1,0)$ supergravity plus anti-selfdual supermultiplet $(g_{\mu\nu}, B_{\mu\nu}, \varphi, \psi^A_{\mu+}, \chi^A_-)$, Yang-Mills multiplet (A_μ, λ^A_+) and hypermultiplet (ϕ^α, ψ^a_-). All the spinors are symplectic Majorana-Weyl, $A = 1,2$ label the doublet of the R symmetry group $\mathrm{Sp}(1)_R$ and $a = 1,\ldots,2n_H$ labels the fundamental representation of $\mathrm{Sp}(n_H)$. The chiralities of the fermions are denoted by $\pm$.

The hyperscalars ϕ^α, $\alpha = 1,\ldots,4n_H$ parameterize the coset $\mathrm{Sp}(n_H,1)/\,\mathrm{Sp}(n_H) \otimes \mathrm{Sp}(1)_R$. This choice is due to its notational simplicity. Our formulae can straightforwardly be adapted to more general quaternionic coset spaces G/H whose list can be found, for example in [37]. In this paper, we gauge the group

$$K \times \mathrm{Sp}(1)_R \subset \mathrm{Sp}(n_H,1)\,, \qquad K \subseteq \mathrm{Sp}(n_H)\ . \tag{2.1}$$

The group K is taken to be semi-simple, and the $\mathrm{Sp}(1)_R$ part of the gauge group can easily be replaced by its $\mathrm{U}(1)_R$ subgroup.

We proceed by defining the basic building blocks of the model constructed in [2] in an alternative notation. The vielbein V_α^{aA}, the $\mathrm{Sp}(n_H)$ composite connection Q_α^{ab} and the $\mathrm{Sp}(1)_R$ composite connection Q_α^{AB} on the coset are defined via the Maurer-Cartan form as

$$L^{-1}\partial_\alpha L = V_\alpha^{aA} T_{aA} + \tfrac{1}{2}\, Q_\alpha^{ab} T_{ab} + \tfrac{1}{2}\, Q_\alpha^{AB} T_{AB}\,, \tag{2.2}$$

where L is the coset representative, $(T_{ab}, T_{AB}, iT_{aA}) \equiv T_{\widehat{A}\widehat{B}}$ obey the $\mathrm{Sp}(n_H,1)$ algebra

$$\begin{aligned}
[T_{\widehat{A}\widehat{B}}, T_{\widehat{C}\widehat{D}}] &= -\Omega_{\widehat{B}\widehat{C}} T_{\widehat{A}\widehat{D}} - \Omega_{\widehat{A}\widehat{C}} T_{\widehat{B}\widehat{D}} - \Omega_{\widehat{B}\widehat{D}} T_{\widehat{A}\widehat{C}} - \Omega_{\widehat{A}\widehat{D}} T_{\widehat{B}\widehat{C}}\,,\\
\Omega_{\widehat{A}\widehat{B}} &= \begin{pmatrix} \epsilon_{AB} & 0 \\ 0 & \Omega_{ab} \end{pmatrix}\,.
\end{aligned} \tag{2.3}$$

The generator T_{aA} is hermitian and (T_{AB}, T_{ab}) are anti-hermitian. The vielbeins obey the following relations:

$$g_{\alpha\beta} V^\alpha_{aA} V^\beta_{bB} = \Omega_{ab}\epsilon_{AB}\,, \qquad V^\alpha_{aA} V^{\beta aB} + \alpha \leftrightarrow \beta = g^{\alpha\beta}\delta^B_A\,, \tag{2.4}$$

where $g_{\alpha\beta}$ is the metric on the coset. Another useful definition is that of the three quaternionic Kahler structures given by

$$V^A_{\alpha a} V^{aB}_\beta - A \leftrightarrow B = 2J^{AB}_{\alpha\beta}\ . \tag{2.5}$$

Next, we define the components of the gauged Maurer-Cartan form as

$$L^{-1}D_\mu L = P_\mu^{aA}T_{aA} + \tfrac{1}{2}\,Q_\mu^{ab}T_{ab} + \tfrac{1}{2}\,Q_\mu^{AB}T_{AB}\,, \tag{2.6}$$

where

$$D_\mu L = \left(\partial_\mu - A_\mu^I T^I\right) L\ , \tag{2.7}$$

A_μ^I are the gauge fields of $K \times \mathrm{Sp}(1)_R$. All gauge coupling constants are set equal to unity for simplicity in notation. They can straightforwardly be re-instated. We also use the notation

$$T^I = (T^{I'}, T^r)\,, \qquad T_r = 2T_r^{AB}\,T_{AB}\,, \qquad T_{AB}^r = -\tfrac{i}{2}\,\sigma_{AB}^r\,, \quad r = 1,2,3\ . \tag{2.8}$$

The components of the Maurer-Cartan form can be expressed in terms of the covariant derivative of the scalar fields as follows [38]

$$P_\mu^{aA} = (D_\mu\phi^\alpha)V_\alpha^{aA}\,, \qquad Q_\mu^{ab} = (D_\mu\phi^\alpha)Q_\alpha^{ab} - A_\mu^{ab}\,, \qquad Q_\mu^{AB} = (D_\mu\phi^\alpha)Q_\alpha^{AB} - A_\mu^{AB}\,, \tag{2.9}$$

where

$$D_\mu\phi^\alpha = \partial_\mu\phi^\alpha - A_\mu^I K^{I\alpha}\,, \tag{2.10}$$

and $K^I(\phi)$ are the Killing vectors that generate the $K \times \mathrm{Sp}(1)_R$ transformations on G/H.

Other building blocks to define the model are certain C-functions on the coset. These were defined in [3], and studied further in [38] where it was shown that they can be expressed as

$$L^{-1}T^I L \equiv C^I = C^{IaA}T_{aA} + \tfrac{1}{2}C^{IAB}T_{AB} + \tfrac{1}{2}C^{Iab}T_{ab}\,. \tag{2.11}$$

Differentiating and using the algebra (2.3) gives the useful relation

$$D_\mu C^I = \left(P^a_{\mu\,B}C^{IAB} + P_{\mu b}{}^A C^{Iab}\right) T_{aA} + P_\mu^{aA}C_a^{IB}\,T_{AB} + P_\mu^{aA}C^{Ib}{}_A\,T_{ab}\ . \tag{2.12}$$

Moreover, using (2.6) and (2.9) we learn that

$$K^{I\alpha}V_\alpha^{aA} = C^{IaA}\,, \qquad K^{I\alpha}Q_\alpha^{ab} = C^{Iab} - \delta^{II'}T_{I'}^{ab}\,, \qquad K^{I\alpha}Q_\alpha^{AB} = C^{IAB} - \delta^{Ir}\,T_r^{AB}\ . \tag{2.13}$$

Finally, it is straightforward and useful to derive the identities

$$D_{[\mu}P_{\nu]}^{aA} = -\tfrac{1}{2}\,F_{\mu\nu}^I C^{IaA}\ , \tag{2.14}$$

$$P_{[\mu}^{aA}P_{\nu]A}^b = \tfrac{1}{2}\,Q_{\mu\nu}^{ab} + \tfrac{1}{2}F_{\mu\nu}^I C^{Iab}\,, \tag{2.15}$$

$$P_{[\mu}^{aA}P_{\nu]a}{}^B = \tfrac{1}{2}\,Q_{\mu\nu}^{AB} + \tfrac{1}{2}F_{\mu\nu}^I C^{IAB}\ . \tag{2.16}$$

2.2 Field equations and supersymmetry transformation rules

The Lagrangian for the anomaly free model we are studying can be obtained from [2] or [3]. We shall use the latter in the absence of Lorentz Chern-Simons terms and Green-Schwarz anomaly counterterms. Thus, the bosonic sector of the Lagrangian is given by [3]

$$e^{-1}\mathcal{L} = R - \tfrac{1}{4}(\partial\varphi)^2 - \tfrac{1}{12}e^{\varphi}\,G_{\mu\nu\rho}G^{\mu\nu\rho} - \tfrac{1}{4}\,e^{\frac{1}{2}\varphi}\,F_{\mu\nu}^I\,F^{I\mu\nu} - 2P_\mu^{aA}\,P_{aA}^\mu - 4\,e^{-\frac{1}{2}\varphi}\,C_{AB}^I C^{IAB}\,, \tag{2.17}$$

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where the Yang-Mills field strength is defined by $F^I = dA^I + \frac{1}{2} f^{IJK} A^J \wedge A^K$ and G obeys the Bianchi identity

$$dG = \tfrac{1}{2} F^I \wedge F^I \ . \tag{2.18}$$

The bosonic field equations following from the above Lagrangian are [3]

$$\begin{aligned}
R_{\mu\nu} &= \tfrac{1}{4}\partial_\mu\varphi\,\partial_\nu\varphi + \tfrac{1}{2}e^{\frac{1}{2}\varphi}\,(F^2_{\mu\nu} - \tfrac{1}{8}F^2\,g_{\mu\nu}) + \tfrac{1}{4}e^{\varphi}\,(G^2_{\mu\nu} - \tfrac{1}{6}G^2\,g_{\mu\nu}) \\
&\quad -2P_\mu^{aA}P_{\nu aA} + e^{-\frac{1}{2}\varphi}(C^I_{AB}C^{IAB})\,g_{\mu\nu}\ , \\
\Box\,\varphi &= \tfrac{1}{4}e^{\frac{1}{2}\varphi}\,F^2 + \tfrac{1}{6}e^{\varphi}\,G^2 - 4\,e^{-\frac{1}{2}\varphi}\,C^I_{AB}C^{IAB} \\
D_\rho(e^{\frac{1}{2}\varphi}\,F^{I\rho}{}_\mu) &= \tfrac{1}{2}e^{\varphi}\,F^{I\rho\sigma}G_{\rho\sigma\mu} + 4P_\mu^{aA}C^I_{aA}\,, \\
\nabla_\rho\,(e^{\varphi}\,G^\rho{}_{\mu\nu}) &= 0\,, \\
D_\mu P^{\mu aA} &= 4e^{-\frac{1}{2}\varphi}C^{IAB}C^{Ia}{}_B\,,
\end{aligned} \tag{2.19}$$

where we have used a notation $V^2_{\mu\nu} = V_{\mu\lambda_2\ldots\lambda_p}V_\nu{}^{\lambda_2\ldots\lambda_p}$ and $V^2 = g^{\mu\nu}V_{\mu\nu}$ for a p-form V, and $F^2 = F^I_{\mu\nu}F^{\mu\nu I}$. The local supersymmetry transformations of the fermions, up to cubic fermion terms that will not effect our results for the Killing spinors, are given by [3]

$$\delta\psi_\mu = D_\mu\varepsilon + \tfrac{1}{48}e^{\frac{1}{2}\varphi}G^+_{\nu\sigma\rho}\,\Gamma^{\nu\sigma\rho}\,\Gamma_\mu\,\varepsilon\,, \tag{2.20}$$

$$\delta\chi = \tfrac{1}{4}\left(\Gamma^\mu\partial_\mu\varphi - \tfrac{1}{6}e^{\frac{1}{2}\varphi}G^-_{\mu\nu\rho}\,\Gamma^{\mu\nu\rho}\right)\varepsilon\,, \tag{2.21}$$

$$\delta\lambda^I_A = -\tfrac{1}{8}F^I_{\mu\nu}\Gamma^{\mu\nu}\varepsilon_A - e^{-\frac{1}{2}\varphi}C^I_{AB}\,\varepsilon^B\,, \tag{2.22}$$

$$\delta\psi^a = P_\mu^{aA}\Gamma^\mu\varepsilon_A\,, \tag{2.23}$$

where $D_\mu\varepsilon_A = \nabla_\mu\varepsilon_A + Q_{\mu A}{}^B\varepsilon_B$, with ∇_μ containing the standard torsion-free Lorentz connection only. The transformation rules for the gauge fermions differ from those in [2] by a field redefinition.

3. Killing spinor conditions

The Killing spinor in the present context is defined to be the spinor of the supersymmetry transformations which satisfies the vanishing of the supersymmetric variations of all the spinors in the model. The well known advantage of seeking such spinors is that the necessary and sufficient conditions for their existence are first order equations which are much easier than the second order field equations, and moreover, once they are solved, the integrability conditions for their existence can be shown to imply most of the field equations automatically. In deriving the necessary and sufficient conditions for the existence of Killing spinors, it is convenient to begin with the construction of the nonvanishing fermionic bilinears, which provide a convenient tool for analyzing these conditions. In this section, firstly the construction and analysis of the fermionic bilinears are given, and then all the necessary and sufficient conditions for the existence of Killing spinor are derived.

3.1 Fermionic bilinears and their algebraic properties

There are only two nonvanishing fermionic bilinears that can be constructed from *commuting* symplectic-Majorana spinor ϵ^A. These are:

$$\bar{\epsilon}^A\Gamma_\mu\epsilon^B \equiv V_\mu\epsilon^{AB}\,,$$
$$\bar{\epsilon}^A\Gamma_{\mu\nu\rho}\epsilon^B \equiv X^r_{\mu\nu\rho}T_r^{AB}\ . \tag{3.1}$$

Note that X^r is a self-dual three-form due to chirality properties. From the Fierz identity $\Gamma_{\mu(\alpha\beta}\Gamma^\mu{}_{\gamma)\delta} = 0$, it follows that

$$V^\mu V_\mu = 0\,, \qquad i_V X^r = 0\ . \tag{3.2}$$

Introducing the orthonormal basis

$$ds^2 = 2e^+e^- + e^ie^i\,, \tag{3.3}$$

and identifying

$$e^+ = V\,, \tag{3.4}$$

the equation $i_V X^r = 0$ and self-duality of X^r yield

$$X^r = 2V \wedge I^r\,, \tag{3.5}$$

where

$$I^r = \tfrac{1}{2}I^r_{ij}\,e^i \wedge e^j \tag{3.6}$$

is anti-self dual in the 4-dimensional metric $ds_4^2 = e^ie^i$. Straightforward manipulations involving Fierz identities imply that I^r are quaternionic structures obeying the defining relation

$$(I^r)^i{}_k\,(I^s)^k{}_j = \epsilon^{rst}(I^t)^i{}_j - \delta^{rs}\delta^i_j\ . \tag{3.7}$$

Finally, using the Fierz identity $\Gamma_{\mu(\alpha\beta}\Gamma^\mu{}_{\gamma)\delta} = 0$ once more, one finds that

$$V_\mu\Gamma^\mu\epsilon = \Gamma^+\epsilon = 0\ . \tag{3.8}$$

If there exists more than one linearly independent Killing spinor, one can construct as many linearly independent null vectors. In this case (3.8) is obeyed by each Killing spinor and the corresponding null vector, i.e. $V^1_\mu\Gamma^\mu\epsilon_1 = 0$, $V^2_\mu\Gamma^\mu\epsilon_2 = 0$, but it may be that $V^1_\mu\Gamma^\mu\epsilon_2 \neq 0$ and/or $V^2_\mu\Gamma^\mu\epsilon_1 \neq 0$. In that case, (3.8) should be relaxed since ϵ should be considered as a linear combination of ϵ_1 and ϵ_2.

3.2 Conditions from $\delta\lambda^I = 0$

Multiplying (2.22) with $\bar{\epsilon}^B\Gamma^\rho$, we obtain

$$i_V F^I = 0\,, \tag{3.9}$$
$$F^{Iij}I^r_{ij} = 4e^{-\frac{1}{2}\varphi}\,C^{Ir}\ . \tag{3.10}$$

The second has been simplified by making use of (3.9) and (3.5). Multiplying (2.22) with $\bar{\epsilon}^B \Gamma_{\lambda\tau\rho}$, on the other hand, gives

$$F^I \wedge V + \star(F^I \wedge V) + 2e^{\frac{1}{2}\varphi}\, C^{Ir} X^r = 0\,, \tag{3.11}$$

$$\tfrac{3}{4} F^{I\sigma}{}_{[\mu} X^r_{\nu\rho]\sigma} + \tfrac{1}{2}\epsilon^{rst} e^{-\frac{1}{2}\varphi} C^{Is} X^t_{\mu\nu\rho} = 0\ . \tag{3.12}$$

One can show that these two equations are identically satisfied upon the use of (3.9) and (3.10), which, in turn imply that F must take the form

$$F^I = -e^{-\frac{1}{2}\varphi}\, C^{Ir} I^r + \widetilde{F}^I + V \wedge \omega^I\,, \tag{3.13}$$

where $\widetilde{F}^I = \frac{1}{2}\widetilde{F}^I_{ij}\, e^i \wedge e^j$ is self-dual, and $\omega^I = \omega^I_i\, e^i$. Reinstating the gauge coupling constants, we note that the C-function dependent term will be absent when the index I points in the direction of a subgroup of $K \subset \mathrm{Sp}(2n_H)$ under which all the hyperscalars are neutral.

Substituting (3.13) into the supersymmetry transformation rule, and recalling (3.8), one finds that (2.22) gives the additional conditions on the Killing spinor

$$\left(\tfrac{1}{8} I^r_{ij}\Gamma^{ij}\delta^A_B - T^{rA}{}_B\right)\, \epsilon^B = 0\ . \tag{3.14}$$

The contribution from $\widetilde{F}$ drops out due to chirality-duality properties involved. Writing this equation as $\mathcal{O}^r\epsilon = 0$, one can check that $[\mathcal{O}^r, \mathcal{O}^s] = \epsilon^{rst}\mathcal{O}^t$. Thus, any two projection imply the third one.

In summary, the necessary and sufficient conditions for $\delta\lambda^I = 0$ are (3.13) and (3.14).

3.3 Conditions from $\delta\psi^a = 0$

This time multiplying (2.22) with $\bar{\epsilon}^B$ and $\bar{\epsilon}^B\Gamma_{\lambda\tau}$ gives rise to four equations which can be shown to imply

$$V^\mu P^{aA}_\mu = 0\,, \tag{3.15}$$

$$P^{aA}_i = 2(I^r)_i{}^j\, (T^r)^A{}_B\, P^{aB}_j\ . \tag{3.16}$$

Using (2.5) and (2.9), we can equivalently reexpress the second equation above as

$$D_i\phi^\alpha = (I^r)_i{}^j\, (J^r)_\beta{}^\alpha\, D_j\phi^\beta\ . \tag{3.17}$$

Writing (3.16) as $P^a = \mathcal{O}P^a$, we find that $(\mathcal{O}-1)(\mathcal{O}-3) = 0$. Thus, (3.16) implies that P^a is an eigenvector of $\mathcal{O}$ with eigenvalue one. Moreover, using (3.16) directly in the supersymmetry transformation rule (2.23), and using the projection condition (3.14), we find that $\delta\psi^a = 3\delta\psi^a$, and hence vanishing.

In summary, the necessary and sufficient conditions for $\delta\psi^a = 0$ are (3.15), (3.16) (or equivalently (3.17)), together with the projection condition (3.14).

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3.4 Conditions from $\delta\chi = 0$

The analysis for this case is identical to that given in [18], so we will skip the details, referring to this paper. Multiplying (2.21) with $\bar{\epsilon}^B$ and $\bar{\epsilon}^B\Gamma_{\lambda\tau}$ gives four equations which can be satisfied by

$$V^\mu\partial_\mu\varphi = 0\,, \tag{3.18}$$

and parametrizing G^- as

$$e^{\frac{1}{2}\varphi}\,G^- = \tfrac{1}{2}(1-\star)\left[V\wedge e^-\wedge d\varphi + V\wedge K\right]\,, \tag{3.19}$$

where $\star$ is the Hodge-dual, $K = \frac{1}{2}K_{ij}\,e^i\wedge e^j$ is self-dual. In fact, these two conditions are the necessary and sufficient conditions for satisfying $\delta\chi = 0$.

3.5 Conditions from $\delta\psi_\mu = 0$

Multiplying (2.20) with $\bar{\epsilon}\Gamma_\nu$, we find

$$\nabla_\mu V_\nu = -\tfrac{1}{2}e^{\frac{1}{2}\varphi}\,G^+_{\mu\nu\rho}V^\rho\,, \tag{3.20}$$

which implies that V^μ is a Killing vector. Similarly, multiplying (2.20) with $\bar{\epsilon}\Gamma_{\nu\rho\sigma}$ gives an expression for $\nabla_\sigma X^r_{\mu\nu\rho}$. Using (3.20) one finds that this expression is equivalent to

$$D_\mu I^r_{ij} = e^{\frac{1}{2}\varphi}G^{+k}{}_{\mu[i}\,I^r_{j]k}\,, \tag{3.21}$$

where $D_\mu I^r \equiv \nabla_\mu I^r + \epsilon^{rst}Q^s_\mu I^t$. One can use (3.21) to fix the composite $\mathrm{Sp}(1)_R$ connection as follows

$$Q^r_\mu = \tfrac{1}{4}e^\varphi G^{(+)}_{\mu ij}I^{rij} - \tfrac{1}{8}\epsilon^{rst}I^{sij}\nabla_\mu I^t_{ij}\,. \tag{3.22}$$

Manipulations similar to those in [18] shows that, using (3.14) and (3.20), the variation $\delta\psi_\mu = 0$ is directly satisfied, with ϵ constant, in a frame where I^r_{ij} are constants.

In summary, the necessary and sufficient conditions for $\delta\psi_\mu = 0$ are (3.20), (3.21), together with the projection condition (3.14).

4. Integrability conditions for the existence of a Killing spinor

Assuming the Killing spinor conditions derived in the previous section, the attendant integrability conditions can be used to show that certain field equations are automatically satisfied. Since the field equations are complicated second order equations, it is therefore convenient to determine those which follow from the integrability, and identify the remaining equations that need to be satisfied over and above the Killing spinor conditions.

Let us begin by introducing the notation

$$\delta\psi_\mu = \widetilde{D}_\mu\epsilon\,,\qquad \delta\chi = \tfrac{1}{4}\Delta\epsilon\,,\qquad \delta\lambda^I = e^{-\frac{1}{2}\varphi}\Delta^I\epsilon\,,\qquad \delta\psi^a = \Delta^{aA}\epsilon_A\,, \tag{4.1}$$

for the supersymmetry variations and

$$R_{\mu\nu} = J_{\mu\nu}\,,\quad \Box\varphi = J\,,\quad D_\mu(e^{\frac{1}{2}\varphi}F^{I\mu\nu}) = J^{I\nu}\,,\quad D_\mu P^{\mu aA} = J^{aA}\,, \tag{4.2}$$

for bosonic field equations. Then we find that

$$
\begin{aligned}
\Gamma^\mu[\widetilde{D}_\mu, \Delta^I]\epsilon^A &= 2\left[D_\mu(e^{\frac{1}{2}\varphi}F^{I\mu\nu}) - J^{I\nu}\right]\Gamma_\nu\epsilon^A \\
&\quad + e^{\frac{1}{2}\varphi}\left(D_\mu F^I_{\nu\rho}\right)\Gamma^{\mu\nu\rho}\epsilon^A - 8\Gamma^\mu\left(D_\mu C^{IAB} + 2C^{Ia(A}P_{\mu a}{}^{B)}\right)\epsilon_B \\
&\quad - 2[\Delta, \Delta^I]\epsilon^A + 2e^{\frac{1}{2}\varphi}F^I_{\mu\nu}\Gamma^{\mu\nu}\left(\delta\chi^A\right) + 16C^{IaA}\left(\delta\psi_a\right), \\
&\quad + 8e^{\frac{1}{2}\varphi}f^{IJK}A^J_\mu\Gamma^\mu(\delta\lambda^{KA}),
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
\Gamma^\mu[\widetilde{D}_\mu, \Delta^{aA}]\epsilon_A &= \left(D_\mu P^{\mu aA} - J^{aA}\right)\epsilon_A \\
&\quad + \Gamma^{\mu\nu}\left(D_\mu P^{aA}_\nu - \tfrac{1}{2}F^I_{\mu\nu}C^{IaA}\right)\epsilon_A \\
&\quad - 4C^{IaA}(\delta\lambda^I_A) - \tfrac{1}{24}e^{\frac{1}{2}\varphi}G_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\left(\delta\psi^a\right),
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
\Gamma^\mu[\widetilde{D}_\mu, \Delta]\epsilon_A &= \left(\Box\varphi - J\right)\epsilon_A - \tfrac{1}{2}e^{-\frac{1}{2}\varphi}D_\mu(e^\varphi G^\mu{}_{\nu\rho})\Gamma^{\nu\rho}\epsilon_A \\
&\quad - \tfrac{1}{6}e^{\frac{1}{2}\varphi}\Gamma^{\mu\nu\rho\sigma}\left(\nabla_\mu G_{\nu\rho\sigma} - \tfrac{3}{4}F^I_{\mu\nu}F^I_{\rho\sigma}\right)\epsilon_A \\
&\quad - \left(e^{\frac{1}{2}\varphi}F^I_{\mu\nu}\Gamma^{\mu\nu}\epsilon_{AB} + 8C^I_{AB}\right)\delta\lambda^{IB} + \tfrac{1}{6}e^{\frac{1}{2}\varphi}G_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\left(\delta\chi_A\right),
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
\Gamma^\nu[\widetilde{D}_\mu, \widetilde{D}_\nu]\epsilon^A &= \tfrac{1}{2}\left(R_{\mu\nu} - J_{\mu\nu}\right)\Gamma^\nu\epsilon^A + \tfrac{1}{16}e^{-\frac{1}{2}\varphi}\nabla^\nu(e^\varphi G_{\nu\rho\sigma})\Gamma^{\rho\sigma}\Gamma_\mu\epsilon^A \\
&\quad + \tfrac{1}{48}e^{\frac{1}{2}\varphi}\Gamma^{\rho\sigma\lambda\tau}\Gamma_\mu\left(\nabla_\rho G_{\sigma\lambda\tau} - \tfrac{3}{4}F^I_{\rho\sigma}F^I_{\lambda\tau}\right)\epsilon^A \\
&\quad + \left(Q^{AB}_{\mu\nu} + F^I_{\mu\nu}C^{IAB} - 2P^{aA}_{[\mu}P_{\nu]a}{}^B\right)\Gamma^\nu\epsilon_B \\
&\quad + \tfrac{1}{2}\left[\partial_\mu\varphi + \tfrac{1}{12}e^{\frac{1}{2}\varphi}G_{\nu\rho\sigma}\Gamma^{\nu\rho\sigma}\Gamma_\mu\right]\delta\chi^A + 2P^{aA}_\mu(\delta\psi_a) \\
&\quad - \tfrac{1}{8}e^{\frac{1}{2}\varphi}\left[(\Gamma^{\nu\rho}\Gamma_\mu - 4\delta^\nu_\mu\Gamma^\rho)F^I_{\nu\rho}\epsilon^{AB} - \Gamma_\mu C^{IAB}\right]\delta\lambda^I_B\,.
\end{aligned} \tag{4.6}
$$

If one makes the ansatz for the potentials directly , then the Bianchi identities and the relations (2.12) and (2.14)–(2.16) are automatically satisfied. Otherwise, all of these equations must be checked. Assuming that these are satisfied, from (4.3) it follows that the Yang-Mills field equation $K_\mu = 0$, *except for* $K_+ = 0$, is automatically satisfied, as can be seen by multiplying $K_\mu\Gamma^\mu\epsilon^A = 0$ by $\bar\epsilon^B$ and $K_\nu\Gamma^\nu$, recalling $\Gamma^+\epsilon = 0$ and further simple manipulations. Similarly, from (4.4) it follows that the hyperscalar field equation $K^{aA} = 0$ is automatically satisfied as can be seen by multiplying $K^{aA}\epsilon_A = 0$ by $\bar\epsilon_B\Gamma^\mu$. Finally, from (4.5) and (4.6), it follows that the dilaton and Einstein equation $E_{\mu\nu} = 0$, *except* $E_{++} = 0$, are automatically satisfied, provided that we also impose the G-field equation. This can be seen by multiplying $E_{\mu\nu}\Gamma^\nu\epsilon_A = 0$ with $\bar\epsilon_B$ and $E_{\mu\rho}\Gamma^\rho$ and simply manipulations that make use of $\Gamma^+\epsilon = 0$.

In summary, once the Killing spinor conditions are obeyed, all the field equations are automatically satisfied as well, except the following,

$$R_{++} = J_{++}\,, \qquad D_\mu(e^{\frac{1}{2}\varphi}F^{I\mu}{}_+) = J^I_+\,, \qquad D_\mu(e^{\varphi}G^{\mu\nu\rho}) = 0\,, \tag{4.7}$$

and the Bianchi identities $DF^I = 0$ and $dG = \frac{1}{2}F^I \wedge F^I$.

It is useful to note that in the case of gravity coupled to a non-linear sigma model, the scalar field equation follows from the Einstein's equation and the contracted Bianchi identity only when the scalar map is a submersion (i.e. when the rank of the matrix $\partial_\mu\phi^\alpha$ is equal to the dimension of the scalar manifold). In our model, however, the scalar field equation is automatically satisfied as a consequence of the Killing spinor integrability conditions, without having to impose such requirements. This is all the more remarkable given the fact that there are contributions to the energy-momentum tensor from fields other than the scalars.

Finally, in analyzing the set of equations summarized above for finding a supersymmetric solution, it is convenient to parametrize the metric, which admits a null Killing vector, in general as [17]

$$ds^2 = 2H^{-1}(du+\beta)\left(dv + \omega + \frac{\mathcal{F}}{2}(du+\beta)\right) + H ds^2_B\,, \tag{4.8}$$

with

$$\begin{aligned} e^+ &= H^{-1}(du+\beta)\,, \\ e^- &= dv + \omega + \tfrac{1}{2}\mathcal{F}He^+\,, \\ e^i &= H^{1/2}\tilde{e}_\alpha{}^i dy^\alpha\,, \end{aligned} \tag{4.9}$$

where $ds^2_B = h_{\alpha\beta}dy^\alpha dy^\beta$ is the metric on the base space $\mathcal{B}$, and we have $\beta = \beta_\alpha dy^\alpha$ and $\omega = \omega_\alpha dy^\alpha$ as 1-forms on $\mathcal{B}$. These quantities as well as the functions H and $\mathcal{F}$ depend on u and y but not on v. Now, as in [17], defining the 2-forms on $\mathcal{B}$ by

$$\tilde{J}^r = H^{-1}I^r\,, \tag{4.10}$$

these obey

$$(\tilde{J}^r)^\alpha{}_\gamma\,(\tilde{J}^s)^\gamma{}_\beta = \epsilon^{rst}(\tilde{J}^t)^\alpha{}_\beta - \delta^{rs}\delta^\alpha_\beta\,, \tag{4.11}$$

where raising and lowering of the indices is understood to be made with $h_{\alpha\beta}$. Note that the index $\alpha = 1,\ldots,4$ labels the coordinates y^α on the base space $\mathcal{B}$. This should not be confused with the index $\alpha = 1,\ldots,n_H$ that labels the coordinates ϕ^α of the scalar manifold!

A geometrically significant equation satisfied by $\tilde{J}^r$ can be obtained from (3.21), and with the help of (3.20) it takes the form [18],

$$\tilde{\nabla}_i\tilde{J}^r_{jk} + \epsilon^{rst}Q^s_i\tilde{J}^t_{jk} - \beta_i\dot{\tilde{J}}{}^r_{jk} - \dot{\beta}_{[j}\tilde{J}^r_{k]i} + \delta_{i[j}\dot{\beta}^m\tilde{J}^r_{k]m} = 0\;, \tag{4.12}$$

where $\tilde{\nabla}_i$ is the covariant derivative on the base space $\mathcal{B}$ with the metric ds^2_B and $\dot{\beta} \equiv \partial_u\beta$.

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5. The dyonic string solution

For the string solution we shall activate only four hyperscalars, setting all the rest equal to zero. In the quaternionic notation of appendix B, this means

$$t = \begin{pmatrix} \phi \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5.1}$$

In what follows we shall use the map

$$\phi = \phi^{A'A} = \phi^\alpha (\sigma_\alpha)^{A'A} \,, \tag{5.2}$$

where $\sigma_\alpha = (1, -i\vec{\sigma})$ are the constant van der Wardeen symbols for SO(4). Moreover, we shall chose the gauge group K such that

$$T^{I'}\, t = 0 \ . \tag{5.3}$$

This condition can be easily satisfied by taking K to be a subgroup of $\mathrm{Sp}(n_H - 1)$ which evidently leaves t given in (5.1) invariant. Finally, we set

$$A_\mu^{I'} = 0 \ . \tag{5.4}$$

Then, supersymmetry condition (3.13) in I' direction is satisfied by setting $\widetilde{F}^{I'} = 0 = \omega^{I'}$ and noting that $C^{I'r} = 0$ in view of (5.3) (see (B.10)). The supersymmetry condition (3.16) is also satisfied along the directions in which the hyperscalars are set to zero. Therefore, the model effectively reduces to one in which the hyperscalars are described by $\mathrm{Sp}(1,1)/\,\mathrm{Sp}(1)\times \mathrm{Sp}(1)$, which is equivalent to a 4-hyperboloid $H_4 = \mathrm{SO}(4,1)/\,\mathrm{SO}(4)$.

Using (5.2) in the definition of $D_\mu t$ given in (B.8), we obtain

$$D_\mu \phi^\alpha = \partial_\mu \phi^\alpha - \tfrac{1}{2} A_\mu^r (\rho^r)^\alpha{}_\beta\, \phi^\beta \,, \tag{5.5}$$

where the 't Hooft symbols ρ^r are constant matrices defined as

$$\rho^r_{\alpha\beta} = \mathrm{tr}\,(\sigma_\alpha\, T^r\, \bar{\sigma}_\beta) \ . \tag{5.6}$$

These are anti-self dual and their further properties are given in appendix A.

For the metric we choose

$$\beta = 0 \,, \qquad \omega = 0 \,, \qquad \mathcal{F} = 0 \,, \qquad h_{\alpha\beta} = \Omega^2 \delta_{\alpha\beta} \,, \tag{5.7}$$

in the general expression (4.8), so that our ansatz takes the form

$$ds^2 = 2H^{-1}\, du dv + H ds_B^2 \,, \qquad ds_B^2 = \Omega^2 dy^\alpha dy^\beta \delta_{\alpha\beta} \,, \tag{5.8}$$

where Ω is a function of $y^2 \equiv y^\alpha y^\beta \delta_{\alpha\beta}$. We also choose the null basis as

$$e^+ = V = H^{-1} du \,, \qquad e^- = dv \ . \tag{5.9}$$

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Thus, $V^\mu\partial_\mu = \partial/\partial v$. Moreover, in the rest of this section, *we shall take all the fields to be independent of u and v.* Given that $\beta = 0$, it also follows from (4.12) that

$$\tilde{\nabla}_i \tilde{J}^r_{jk} + \epsilon^{rst} Q^s_i \tilde{J}^t_{jk} = 0 \; . \tag{5.10}$$

Next, in the general form of $G^{(-)}$ given in (3.19), we choose

$$K = 0 \; . \tag{5.11}$$

Then, from (3.19) and (3.20) we can compute all the components of G^+ and G^-, which yield for $G = G^+ + G^-$ the result

$$G = e^{-\varphi/2} \left(e^+ \wedge e^- \wedge d\varphi_+ + \star_4 \, d\varphi_- \right) \, , \tag{5.12}$$

where $\star_4$ refers to Hodge dual on the transverse space with metric

$$ds_4^2 = H ds_{\mathcal{B}}^2 \, , \tag{5.13}$$

and we have defined

$$\varphi_\pm \; := \; \pm\tfrac{1}{2}\varphi + \ln H \; . \tag{5.14}$$

Next, we turn to the supersymmetry condition (3.17) in the hyperscalar sector. With our ansatz described so far, it can now be written as

$$D_i \phi^{\underline{\alpha}} = (\tilde{J}^r)_i{}^j \, (J^r)_{\underline{\beta}}{}^{\underline{\alpha}} \, D_j \phi^{\underline{\beta}} \, , \tag{5.15}$$

where

$$D_i \phi^{\underline{\alpha}} \equiv D_i \phi^\alpha \, V_\alpha{}^{\underline{\alpha}} \, , \tag{5.16}$$

and $V_\alpha{}^{\underline{\alpha}}$ is the vielbein on H_4, and the above equations are in the basis

$$\tilde{e}^i = \delta^i_\alpha \, \Omega \, dy^\alpha \, , \tag{5.17}$$

referring to the base space $\mathcal{B}$. We also note that

$$J^r_{\underline{\alpha}\underline{\beta}} = \rho^r_{\alpha\beta} \, \delta^\alpha_{\underline{\alpha}} \, \delta^\beta_{\underline{\beta}} \, , \tag{5.18}$$

which follows from rom (C.2) and (C.3). Recall that the 't Hooft matrices $\rho^r_{\alpha\beta}$ are constants. Next, we choose the components of $\tilde{J}^r_{ij}$ to be constants and make the identification

$$\tilde{J}^r = J^r \; . \tag{5.19}$$

Using the quaternion algebra, we can now rewrite (5.15) as

$$D_i \phi_{\underline{\beta}} = \left(\delta_{i\underline{\alpha}} \delta_{j\underline{\beta}} - \delta_{j\underline{\alpha}} \delta_{i\underline{\beta}} - \epsilon_{ij\underline{\alpha}\underline{\beta}} \right) \, D_j \phi_{\underline{\alpha}} \; . \tag{5.20}$$

Symmetric and antisymmetric parts in i and $\underline{\beta}$ give

$$D_i \phi^i = 0 \, , \qquad \phi^i \equiv \phi^{\underline{\alpha}} \, \delta^i_{\underline{\alpha}} \, , \tag{5.21}$$

$$D_i \phi_j - D_j \phi_i = -\epsilon_{ijk\ell} D_k \phi_\ell \; . \tag{5.22}$$

To solve these equations, we make the ansatz

$$\phi^\alpha = f y^\alpha \,, \qquad A^r_\alpha = g\, \rho^r_{\alpha\beta}\, y^\beta \,, \tag{5.23}$$

where f and g are functions of y^2. This ansatz, in particular, implies that the function ω^r arising in the general form of F^r given in (3.13) vanishes. Assuming that the map ϕ^α is 1-1, one can actually use diffeomorphism invariance to set (at least locally) $f = 1$. However, since we have already fixed the form of the metric as in (5.8), chosen a basis as in (5.17), and identified the components of the quaternionic structures $\tilde{J}^r_{ij}$ referring to this orthonormal basis, the reparametrization invariance has been lost. Therefore it is important to keep the freedom of having an arbitrary function in the map (5.23).

Using (5.23) we find that (5.22) is identically satisfied and (5.21) implies

$$g = \frac{4f'y^2 + 8f}{3fy^2} \,, \tag{5.24}$$

where prime denotes derivative with respect to argument, i.e. y^2. Next, the computation of the Yang-Mills field strength from the potential (5.23) gives the result

$$\begin{aligned}
F^r &= F^{r(+)} + F^{r(-)} \,, \qquad F^{r\pm} = \pm \star_4 F^{r\pm} \,, \\
F^{r(-)}_{\alpha\beta} &= (-2g - g'y^2 + \tfrac{1}{2}g^2y^2)\, \rho^r_{\alpha\beta} \,, \\
F^{r(+)}_{\alpha\beta} &\equiv \widetilde{F}^r_{\alpha\beta} = (2g' + g^2)\left(2y_{[\alpha}y^\delta\, \rho^r_{\beta]\delta} + \tfrac{1}{2}y^2\, \rho^r_{\alpha\beta}\right) \,.
\end{aligned} \tag{5.25}$$

Comparing these results with the general form of F^I given in (3.13), we obtain

$$e^{\varphi_-} = \frac{\eta}{\Omega^2} \,, \tag{5.26}$$

where

$$\eta \equiv \left(g'y^2 + 2g - \tfrac{1}{2}g^2y^2\right)(1 - f^2y^2) \,. \tag{5.27}$$

Here we have used the fact that $C^{r,s} = \delta^{rs}/(1-\phi^2)$ as it follows from the formula (B.9). Finally using the composite connection (C.4) in (5.10) we obtain

$$\frac{\Omega'}{\Omega} = \frac{(2f^2 - g)}{2(1 - f^2y^2)} \,. \tag{5.28}$$

This equation can be integrated with the help of (5.24), yielding

$$\Omega = \frac{b}{y^2}\left(\frac{1 - f^2y^2}{f^2y^2}\right)^{1/3} \,, \tag{5.29}$$

where b is an integration constant. One can now see that all necessary and sufficient conditions for the existence of a Killing spinor on this background are indeed satisfied. As shown in the previous section, the integrability conditions for the existence of a Killing spinor imply all field equations except (4.7) and the Bianchi identities on F^I and G. It is easy to check that (4.7) is identically satisfied by our ansatz, except for the G-field equation.

Furthermore, the Yang-Mills Bianchi identity is trivial since we give the potential. Thus, the only remaining equations to be checked are the G-Bianchi identity and the G-field equation. To this end, it is useful to record the result

$$\frac{\epsilon^{\alpha\beta\gamma\delta}}{\sqrt{g_4}} F^r_{\alpha\beta} F^r_{\gamma\delta} = \frac{16Q'}{y^2 H^2 \Omega^4}\,, \tag{5.30}$$

where g_4 is the determinant of the metric for the line element ds_4^2, and

$$Q \equiv (gy^2)^2(gy^2-3)+c\,, \tag{5.31}$$

where c is an integration constant. Interestingly, this term is proportional to the sum of of F^2 and C^2 terms that arise in the dilaton field equation, up to an overall constant.

We now impose the G-field equation $d(e^{\varphi} \star G) = 0$ and the G-Bianchi identity $dG = \frac{1}{2}F^r \wedge F^r$. The G-field equation gives

$$\Box_4 \varphi_+ + \tfrac{1}{2}\partial_\alpha\, \varphi \partial^\alpha \varphi_+ \;=\; 0\,, \tag{5.32}$$

and the G-Bianchi identity amounts to

$$\Box_4 \varphi_- - \tfrac{1}{2}\partial_\alpha \varphi\, \partial^\alpha \varphi_- \;=\; \frac{-2Q'}{y^2 H^2 \Omega^4}\,, \tag{5.33}$$

where the Laplacian is defined with respect to the metric (5.13). These equations can be integrated once to give

$$\varphi'_+ = \frac{\nu e^{-\varphi}}{(y^2)^2 \eta}\,, \qquad \varphi'_- = \frac{(\lambda - \frac{1}{2}Q)}{(y^2)^2\eta}\,, \tag{5.34}$$

where ν, λ are the integration constants, c has been absorbed into the definition of λ, and (5.26) has been used in the form $H\Omega^2 = \eta e^{\varphi/2}$. These equation can be rewritten as

$$(e^{\varphi_+})' = \frac{\nu}{b^2}\left(\frac{f^2y^2}{1-f^2y^2}\right)^{2/3}\,, \tag{5.35}$$

$$(e^{\varphi_-})' = \frac{\lambda - \frac{1}{2}Q}{b^2}\left(\frac{f^2y^2}{1-f^2y^2}\right)^{2/3}\,, \tag{5.36}$$

by recalling $\varphi = \varphi_+ - \varphi_-$, exploiting (5.26) and using the solution (5.29) for Ω. It is important to observe that the second equation in (5.34), has to be consistent with (5.26). Differentiating the latter and comparing the two expressions, we obtain a third order differential equation for the function f:

$$\eta' - \left(\frac{2f^2-g}{1-f^2y^2}\right)\eta = \frac{\lambda - \frac{1}{2}Q}{(y^2)^2}\,. \tag{5.37}$$

In summary, any solution of this equation for f determines also the functions (φ, H, Ω, g), and therefore fixes the solution completely. This is a highly complicated equation, however, and we do not know its general solution at this time. Nonetheless, it is remarkable that an ansatz of the form

$$f = \frac{a}{y^2}\,, \tag{5.38}$$

with a a constant, which gives $g = 4/(3y^2)$ from (5.24), does solve (5.37), and moreover, it fixes the integration constant

$$\lambda = -\tfrac{4}{3}\ . \tag{5.39}$$

Furthermore, it follows from (5.29), (5.26), (5.27) and (5.35) that

$$\Omega = \frac{b}{y^2}\,h^{1/3}\,, \qquad e^{\varphi_-} = \left(\frac{2a}{3b}\right)^2 h^{1/3}\,, \qquad e^{\varphi_+} = 3\nu\left(\frac{a}{b}\right)^2 h^{1/3} + \nu_0\,, \tag{5.40}$$

where ν_0 is an integration constant and

$$h \equiv \frac{y^2}{a^2} - 1\ . \tag{5.41}$$

Thus, the full solution takes the form

$$ds^2 = e^{-\frac{1}{2}\varphi_+}e^{-\frac{1}{2}\varphi_-}(-dt^2 + dx^2) + e^{\frac{1}{2}\varphi_+}e^{\frac{1}{2}\varphi_-}\left(\frac{b}{y^2}\right)^2 h^{2/3}\,dy^\alpha dy^\beta\,\delta_{\alpha\beta}\,, \tag{5.42}$$

$$e^\varphi = e^{\varphi_+}/e^{\varphi_-}\,, \qquad \phi^\alpha = \frac{ay^\alpha}{y^2}\,, \tag{5.43}$$

$$A^r_\alpha = \frac{4}{3y^2}\,\rho^r_{\alpha\beta}y^\beta\,, \tag{5.44}$$

$$G_{\alpha\beta\gamma} = \frac{8}{27(y^2)^2}\,\epsilon_{\alpha\beta\gamma\delta}\,y^\delta\,, \qquad G_{+-\alpha} = -\partial_\alpha e^{-\varphi_+}\,, \tag{5.45}$$

with $\varphi_\pm$ given in (5.40). The form of h dictates that $a^2 < y^2 < \infty$, covering outside of a disk of radius a. The hyperscalars map this region into H^4 which can be viewed as the interior of the disk defined by $\phi^2 < 1$. These scalars are gravitating in the sense that their contribution to the energy momentum tensor, which takes the form $(\mathrm{tr}P_iP_j - \frac{1}{2}g_{ij}\mathrm{tr}P^2)$, does not vanish since the solution gives

$$P_i^{A'A} = \frac{a}{3y^2\left(1-\frac{a^2}{y^2}\right)}\left(\delta_i^\alpha - 4\frac{y_iy^\alpha}{y^2}\right)\sigma_\alpha^{A'A}\ . \tag{5.46}$$

It is possible to apply a coordinate transformation and map the base space into the disc by defining

$$z^\alpha \equiv \frac{ay^\alpha}{y^2}. \tag{5.47}$$

In z^α coordinates the solution becomes

$$ds^2 = e^{-\frac{1}{2}\varphi_+}e^{-\frac{1}{2}\varphi_-}(-dt^2 + dx^2) + L^2 e^{\frac{1}{2}\varphi_+}e^{\frac{1}{2}\varphi_-}\,h^{2/3}\,(dr^2 + r^2 d\Omega_3^2) \tag{5.48}$$

$$e^\varphi = e^{\varphi_+}/e^{\varphi_-}\,, \tag{5.49}$$

$$G = \tfrac{8}{27}\,\Omega_3 - dt \wedge dx \wedge de^{-\varphi_+}\,, \tag{5.50}$$

$$A^r = \tfrac{2}{3}\,r^2 \sigma^r_R\,, \tag{5.51}$$

$$\phi^\alpha = z^\alpha\,, \tag{5.52}$$

where

$$r = \sqrt{z^\alpha z^\beta \delta_{\alpha\beta}}\,, \qquad \Omega_3 = \sigma^1_R \wedge \sigma^2_R \wedge \sigma^3_R\,, \qquad h = \frac{1}{r^2} - 1\,, \tag{5.53}$$

$$e^{\varphi_+} = \frac{3\nu h^{1/3}}{L^2} + \nu_0\,, \qquad e^{\varphi_-} = \frac{4h^{1/3}}{9L^2}\,, \tag{5.54}$$

and $L \equiv b/a$. Here, σ^r_R are the right-invariant one-forms satisfying

$$d\sigma^r_R = \tfrac{1}{2}\epsilon^{rst}\,\sigma^s_R \wedge \sigma^t_R\,, \tag{5.55}$$

and Ω_3 is the volume form on S^3. We have also used the definitions

$$z^\alpha = r\,n^\alpha\,, \qquad n^\alpha n^\beta \delta_{\alpha\beta} = 1\,, \tag{5.56}$$

where dn^α are orthogonal to the unit vectors n^α on the 3-sphere, and satisfy

$$dn^\alpha = \tfrac{1}{2}\rho^{r\alpha}{}_\beta\,\sigma^r_R\,n^\beta\,, \qquad dn^\alpha dn^\beta \delta_{\alpha\beta} = \tfrac{1}{4}d\Omega_3^2\,. \tag{5.57}$$

Given the form of A^r, it is easy to see that the Yang-Mills 2-form $F^r = dA^r - \frac{1}{2}\epsilon^{rst}A^s \wedge A^t$ is not (anti)self-dual, as it is given by

$$F^r = \tfrac{4}{3}\,rdr \wedge \sigma^r_R + \tfrac{1}{3}r^2\left(1 - \tfrac{2}{3}r^2\right)\,\epsilon^{rst}\sigma^s_R \wedge \sigma^t_R\,. \tag{5.58}$$

The field strength $P_i^{A'A}$ on the other hand, takes the form

$$P_i^{A'A} = \frac{1}{1-r^2}\,\left[(1 - \tfrac{2}{3}r^2)\delta^\alpha_i + \tfrac{2}{3}r^2 n_i n^\alpha\right]\,\sigma^{A'A}_\alpha\,. \tag{5.59}$$

We emphasize that, had we started with the identity map $\phi^\alpha = z^\alpha$ from the beginning, the orthonormal basis in which $\tilde{J}^r_{ij}$ are constants would be more complicated than the one given in (5.17). Consequently, (5.28) would change since it uses (5.10) that requires the computation of the spin connection in the new orthonormal basis.

6. Properties of the solution

6.1 Dyonic charges and limits

To begin with, we observe that the solution we have presented above is a dyonic string with with *fixed* magnetic charge given by

$$Q_m = \int_{S^3} G = \frac{8}{27}\,vol_{S^3}\,. \tag{6.1}$$

The electric charge, however, turns out to be proportional to the constant parameter ν as follows:

$$Q_e = \int_{S^3} \star e^{\varphi} G = 2\nu \, vol_{S^3} \ . \tag{6.2}$$

Next, let us compare our solution with that of [21] where a dyonic string solution of the $\mathrm{U}(1)_R$ gauged model in the absence of hypermatter has been obtained. We shall refer to this solution as the GLPS dyonic string. To begin with, the GLPS solution has two harmonic functions with two arbitrary integration constants, as opposed to our single harmonic function h with a fixed and negative integration constant. In our solution, this is essentially due to the fact that we have employed an identity map between a hyperbolic negative constant curvature scalar manifold and space transverse to the string worldsheet.

Next, the transverse space metric ds_4^2 in the GLPS solution is a warped product of a *squashed* 3-sphere with a real line, while in our solution it is conformal to R^4. In GLPS solution the deviation from the round 3-sphere is proportional to a product of $\mathrm{U}(1)_R$ gauge constant and monopole flux due to the $\mathrm{U}(1)_R$ gauge field. Thus, assuming that we are dealing with a gauged theory, the round 3-sphere limit would require the vanishing of the monopole flux, which is not an allowed value in GLPS solution.

As for the 3-form charges, the electric charge is arbitrary in the GLPS as well as our solution. However, while the magnetic charge in the GLPS solution is proportional to $k\xi/g_R$ where k is the monopole flux, g_R is the $\mathrm{U}(1)_R$ coupling constant and ξ is the squashing parameter, and therefore arbitrary, in our solution the magnetic charge is fixed in Planckian units and therefore it is necessarily non-vanishing. This is an interesting property of our solution that results from the interplay between the sigma model manifold whose radius is fixed in units of Plank length, which is typical in supergravities with a sigma model sector, and the four dimensional space transverse to the the string worldsheet.

Our solution has $\mathrm{SO}(1,1) \times \mathrm{SO}(4)$ symmetry corresponding to Poincaré invariance in the string world-sheet and rotational invariance in the transverse space[1]. The metric components exhibit singularities at $r = 0$ and $r = 1$. Too see the coordinate invariant significance of these points, we compute the Ricci scalar as

$$R = \frac{48(\Delta + \mu_0)^2 + \mu_0^2}{r^6 \left(\frac{\Delta}{3\nu}\right)^{\frac{17}{18}} (\Delta + \mu_0)^{\frac{5}{2}}} \ , \tag{6.3}$$

where $\Delta \equiv 3\nu(\frac{1}{r^2} - 1)$ and $\mu_0 \equiv \nu_0 L^2$. We see that, near the boundary $r \to 1$, the Ricci scalar diverges, and there is a genuine singularity there. Since the total volume in the base space is finite, one would expect this singularity can be reached by physical particles at a finite proper time, and we have checked that this is indeed the case. Nonetheless, recall that the boundary is not included in the base space in view of the identity map (5.52) with $\phi^2 = r^2 < 1$. Near the origin r=0, the issue of singularities depends on the parameter ν. If $\nu \neq 0$, then as $r \to 0$ the Ricci scalar approaches the constant value $8/\sqrt{3\nu}$. The metric is

[1]It is clear that if one makes an SO(4) rotation in z^α coordinates, the same transformation should be applied to hyperscalars and 't Hooft symbols $\rho^r_{\alpha\beta}$ to preserve the structure of the solution.

perfectly regular in this limit, and indeed, we find that it takes the form

$$ds^2 \to \frac{L^2}{R_0^2}\, r^{2/3}(-dt^2+dx^2) + \frac{R_0^2 dr^2}{r^2} + R_0^2 d\Omega_3^2\,, \tag{6.4}$$

which is $AdS_3 \times S_3$ with $R_0 = \sqrt{4\nu/3}$. This is to be contrasted with the GLPS solution which approaches the product of AdS_3 with a squashed 3-sphere.

The $r = 0$ point can be viewed as the horizon, and as is usually the case, our solution also has a factor of two enhancement of supersymmetry near the horizon. This is due to the fact that the condition (3.8), which reads $H^{-1}\Gamma^+\epsilon = 0$ has to be relaxed since H^{-1} vanishes in in the $r \to 0$ limit. Note, however, that our solution at generic point has 1/8 supersymmetry to begin with, as opposed to 1/4 supersymmetry of the GLPS solution.

For $\nu = 0$, the $r \to 0$ limit of the metric is

$$ds^2 \to \frac{3L}{2\sqrt{\nu_0}}\, r^{1/3}(-dt^2+dx^2) + \frac{2L\sqrt{\nu_0}}{3}\, r^{-5/3}(dr^2 + r^2 d\Omega_3^2)\,, \tag{6.5}$$

Defining furthermore $du = dr/r^{5/6}$ the metric becomes

$$ds^2 \sim u^2(-dt^2+dx^2+d\Omega_3^2) + du^2. \tag{6.6}$$

Ignoring x and Ω_3 directions, this describes the Rindler wedge which is the near horizon geometry of the Schwarzcshild black hole. The "horizon", which has the topology $R \times \Omega_3$, shrinks to the zero size at $u = 0$ and this gives the singularity in the dyonic string.

Next, consider the boundary limit in which $r \to 1$. First, assuming that $\nu_0 \neq 0$, we find that in the limit $r \to 1$ the metric takes the form

$$ds^2 \sim \frac{1}{u^{1/3}}\left(-dt^2+dx^2+u^4(\,du^2+\frac{1}{u^2}\,d\Omega_3^2)\right) \qquad \text{for} \quad \nu_0 \neq 0\,, \tag{6.7}$$

where we have defined the coordinate $u = h^{1/2}$ and rescaled the string worldsheet coordinates by a constant. For $\nu_0 = 0$, on the other hand, the $r \to 1$ limit of the metric is given by

$$ds^2 \sim \frac{1}{u^{2/3}}\left(-dt^2+dx^2\right) + u^4\left(du^2+\frac{1}{u^2}\,d\Omega_3^2\right) \qquad \text{for} \quad \nu_0 = 0\,, \tag{6.8}$$

where, again, we have defined $u = h^{1/2}$ and rescaled coordinates by constants.

6.2 Coupling of sources

Since the solution involves the harmonic function h, there is also a possibility of a delta function type singularity at the origin since

$$\partial_\alpha \partial^\alpha\, h = -4\pi^2 \delta(\vec{z})\,. \tag{6.9}$$

The presence of such a singularity requires addition of extra sources to supergravity fields to get a proper solution. As it is not known how to write down the coupling of a dyonic string to sources, and as we cannot turn off the magnetic charge, we consider the coupling of the magnetic string to sources. Thus setting $\nu = 0$, from (5.48), (5.49) and (5.52) the dangerous fields that can possibly yield a delta function via (6.9) are the metric, the dilaton ϕ and the three form field G. Indeed from (5.52) we see that

$$dG \sim \delta(\vec{z})\, dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4\,, \tag{6.10}$$

therefore extra (magnetically charged) sources are needed for G at $\vec{z}=0$. For the dilaton we find that the candidate singular term near $\vec{z}=0$ behaves as

$$\Box\varphi \sim z^{11/3}\,\delta(\vec{z}) \to 0\,, \tag{6.11}$$

thus there is no problem at $\vec{z}=0$. Finally for the Ricci tensor expressed in the coordinate basis we find

$$R_{tt} = -R_{xx} \sim z^4\delta(\vec{z}) \to 0\,, \tag{6.12}$$

$$R_{\alpha\beta} \sim z^2\delta(\vec{z})\,\delta_{\alpha\beta} \to 0\ . \tag{6.13}$$

Contracting with the metric one can see that the possible singular part in the Ricci scalar becomes

$$R \sim z^{11/3}\,\delta(\vec{z}) \to 0\,, \tag{6.14}$$

and thus there appears no extra delta function singularity.

The above results can be understood by coupling to supergravity fields a magnetically charged string located at $r=0$ with its action given by

$$S = -\int d^2\sigma e^{\varphi/2}\sqrt{-\gamma} + \int \widetilde{B}\,, \tag{6.15}$$

where γ is the determinant of the induced worldsheet metric and $\widetilde{B}$ is the 2-form potential whose field strength is dual to G. This coupling indeed produces exactly the behavior (6.10) in the Bianchi identity. The source terms in (6.11) and (6.12) are also produced, while the contribution to the right hand side of (6.13) vanishes identically (which does not causes a problem since $z^2\delta(\vec{z})$ vanishes at $z=0$ as well).

6.3 Base space as a tear-drop

The four dimensional base space for our solution (5.48) is

$$\begin{aligned} ds_B^2 &= L^2\left(\frac{1}{r^2}-1\right)^{2/3}\left(dr^2 + r^2 d\Omega_3^2\right) \\ &= \frac{(1-r^2)^{8/3}}{2r^{4/3}}\, ds_{H_4}^2\,, \end{aligned} \tag{6.16}$$

where $ds_{H_4}^2 = 2(dr^2 + r^2 d\Omega_3^2)/(1-r^2)^2$ is the metric on H_4, and its curvature scalar is singular at $r=0$ and $r=1$:

$$R_{\mathcal{B}} = \frac{16}{3L^2}\,\frac{1}{r^2}\,\frac{r^{4/3}}{(1-r^2)^{8/3}}\ . \tag{6.17}$$

Although the overall conformal factor blows at $r = 0$, the total volume of this space turns out to have a finite value $(4\pi^3 L^4)/(9\sqrt{3})$. To that extent, our solution can be viewed as the analog of the Gell-Mann-Zwiebach teardrop solution, though the latter is regular at $r = 0$ as well. The analogy with Gell-Mann-Zwiebach tear-drop is also evident in the fact that the scalar metric has been conformally rescaled by a factor that vanishes at the boundary.

Another tear-drop like feature here is that the base space metric is conformally related to that of H_4 which has negative constant curvature, and that the curvature scalar of the bases space becomes positive due to the conformal factor. This switching of the sign is crucial for satisfying Einstein equation in the internal direction, just as in the case of 2-dimensional Gell-Zwiebach tear-drop.

The base space $\mathcal{B}$ that emerges in the $2 + 4$ split of the $6D$ spacetime is quaternionic manifold, as it admits a quaternionic structure. To decide whther it is Quaternionic Kahler (QK), however, the standard definition that relies on the holonomy group being contained in $\mathrm{Sp}(n) \times \mathrm{Sp}(1) \sim \mathrm{SO}(4)$ becomes vacuous in $4D$ since all $4D$ Riemann manifolds have holonomy group $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$. Nonetheless, there exists a generally accepted and natural definition of QK manifolds in four dimensions, which states that an oriented $4D$ Riemann manifold is QK if the metric is self-dual and Einstein (see [39] for a review). According to this definition, our base space $\mathcal{B}$ is not QK since it is neither self-dual nor Einstein.

6.4 Reduction of metric to five dimensions

Finally, we would like to note the 5-dimensional metric that can be obtained by a Kaluza-Klein reduction along the string direction. The 6-dimensional metric is parametrized in terms of the 5-dimensional metric as

$$ds_6^2 = e^{2\alpha\hat{\phi}} ds_5^2 + e^{2\beta\hat{\phi}} dx^2 \tag{6.18}$$

where $\beta = -3\alpha$ and $\hat{\phi}$ is the Kaluza-Klein scalar. From (5.48) one finds

$$ds_5^2 = -e^{-\frac{2}{3}\varphi_+} e^{-\frac{2}{3}\varphi_-}\, dt^2 + L^2 e^{\frac{1}{3}\varphi_+} e^{\frac{1}{3}\varphi_-} h^{2/3}(dr^2 + d\Omega_3^2), \tag{6.19}$$

where the functions are still given in (5.54).

The metric (6.19) is singular at $r = 0$. For $\nu = 0$ looking at the metric near the singularity one finds

$$ds_5^2 \sim u^2(-dt^2 + d\Omega_3^2) + du^2, \tag{6.20}$$

where $du = dr/r^{7/9}$. The geometry is like the Rindler space but the candidate spherical "horizon" shrinks to zero size at $u = 0$ which produces a singularity. When $\nu \neq 0$, one finds near $r = 0$ that

$$ds_5^2 \sim -r^{8/9} dt^2 + r^{-16/9} dr^2 + r^{2/9} d\Omega_3^2 \tag{6.21}$$

which is again singular at $r = 0$. This singularity is resolved by dimensional *oxidation* which is a well known feature of some black-brane solutions [40].

7. Conclusions

In this paper, we have derived the necessary and sufficient conditions for the existence of a Killing spinor in $N = (1, 0)$, $6D$ gauge supergravity coupled to a single tensor multiplet, vector multiplets and hypermultiplets. This generalizes the analysis of [17] and [18] by the inclusion of the hypermatter. In our case as well, the existence of the Killing spinor implies that the metric admits a null Killing vector. This is in contrast to some other dimensions such as $D = 4, 5$ where time-like and space-like Killing vectors arise in addition to the null one. The Killing spinor existence conditions and their integrability are shown to imply most of the equations of motion. This simplifies greatly the search for exact solutions. The remaining equations to be solved are (i) the Yang-Mills equation in the null direction, (ii) the field equation for the 2-form potential, (iii) the Bianchi identities for the Yang-Mills curvature and the field strength of the 2-form potential, and (iv) the Einstein equation in the double null direction. We parametrize the most general form of a supersymmetric solution which involves a number of undetermined functions. However, we do not write explicitly the equations that these functions must satisfy. These can be straightforwardly derived from the equations just listed.

The existence of a null Killing vector suggests a $2 + 4$ split of spacetime, and search for a string solution, possibly dyonic. Such solutions are already known but none of them involve any active hyperscalar. As a natural application of the general framework presented here, we have then focused on finding a dyonic string solution in which the hyperscalars have been activated.

Indeed, we have found a 1/8 supersymmetric such a dyonic string. The activated scalars parametrize a 4 dimensional submanifold of a quaternionic hyperbolic ball of unit radius, characterized by the coset $\mathrm{Sp}(n_H, 4)/\mathrm{Sp}(n_H) \times \mathrm{Sp}(1)_R$. A key step in the construction of the solution is an identity map between the 4-dimensional scalar submanifold and internal space transverse to the string worldsheet. The spacetime metric turns out to be a warped product of the string worldsheet and a 4-dimensional analog of the Gell-Mann-Zwiebach tear-drop which is noncompact with finite volume. Unlike the Gell-Mann-Zwiebach tear-drop, ours is singular at the origin. There is also a delta function type singularity that comes from the Laplacian acting on a harmonic function present in the solution. This does not present any problem, however, as we place a suitable source which produces contributions to the field equations that balance the delta function terms.

An interesting property of our dyonic string solution is that while its electric charge is arbitrary, its magnetic magnetic charge is fixed in Planckian units, and hence it is necessarily non-vanishing. This interesting feature results from the interplay between the sigma model manifold whose radius is fixed in units of Plank length, as it is the case in almost all supergravities that contain sigma models, and the four dimensional space transverse to the the string worldsheet through the identity map.

The tear-drop is quaternionic but not quaternionic Kahler, since its metric is neither self-dual nor Einstein. The metric is conformally related to that of H_4 which has negative constant curvature, and its curvature scalar becomes positive due to the conformal factor. This switching of the sign is crucial for satisfying Einstein equation in the internal direction, just as in the case of 2-dimensional Gell-Zwiebach tear-drop.

We have also shown to have 1/4 supersymmetric $AdS_3 \times S^3$ near horizon limit where the radii are proportional to the electric charge. This is in contrast with the 1/4 supersymmetric GLPS dyonic string that approaches the product of AdS_3 times a squashed 3-sphere with 1/2 supersymmetry. In GLPS solution the squashing is necessarily non-vanishing for non-vanishing gauge coupling constant, while in our case the round 3-sphere emerges even in presence of nonvanishing gauge coupling.

One might naively expect that a double dimensional reduction of our dyonic string might yield a novel black hole solution in $5D$ with active hyperscalars. However, we find that the resulting $5D$ metric has a naked singularity at the origin.

We conclude with mention of a selected open problems. The existence of the supersymmetric dyonic string solution is encouraging with regard to the string/M theory origin of the $6D$ model. The source couplings we have found may provide additional information towards that end. The existence of black dyonic strings in the $\mathrm{SU}(2)_R$ gauged theory motivates a search for 'naturally' anomaly free such models. We refer the reader to the introduction for what we mean by 'natural'. In any event, the string/M theory of origin of the matter coupled $N = (1, 0)$, $6D$ gauged supergravities remains a challenging open problem.

Here, we have begun to uncover some universal features of supersymmetric solutions in which the sigma models play a nontrivial role. For example, the emergence of tear-drop like metrics in the space transverse to the brane. This is intimately related with another potentially universal mechanism by which a submanifold of the sigma model is identified with the transverse space. One possible generalization might involve more intricate maps from the transverse space to sigma model. It would be useful to find further examples to establish whether the features found here continue to persist in a larger class of supergravity models with supergravity sectors.

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A. Conventions

We use the spacetime signature $(- + + + ++)$ and set $\epsilon^{+-ijkl} = \epsilon^{ijkl}$. We define $\Gamma_7 = \Gamma^{012345}$. The supersymmetry parameter has the positive chirality: $\Gamma_7\,\epsilon = \epsilon$. Thus, $\Gamma_{\mu\nu\rho} = \frac{1}{6}\,\epsilon_{\mu\nu\rho\sigma\lambda\tau}\,\Gamma^{\sigma\lambda\tau}\,\Gamma_7$, and for a self-dual 3-form we have $S_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\epsilon = 0$.

The Hodge-dual of a p-form

$$F = \frac{1}{p!}\,dx^{\mu_1} \wedge \cdots dx^{\mu_p} F_{\mu_1 \ldots \mu_p}\,, \tag{A.1}$$

is calculated using

$$*(dx^{\mu_1} \wedge \cdots dx^{\mu_p}) = \frac{1}{(D-p)!}\, \epsilon_{\nu_1 \ldots \nu_{D-p}}{}^{\mu_1 \ldots \mu_p}\, dx^{\nu_1} \cdots dx^{\nu_{D-p}} \ . \tag{A.2}$$

The 't Hoof symbols are defined as

$$\rho^r_{\alpha\beta} = \mathrm{tr}\,(\sigma_\alpha\, T^r\, \bar\sigma_\beta) \ , \qquad \eta^{r'}_{\alpha\beta} = \mathrm{tr}\,(\bar\sigma_\alpha\, T^{r'}\, \sigma_\beta) \, , \tag{A.3}$$

where $\sigma_\alpha = (1, -i\vec\sigma)$ are the constant van der Wardeen symbols for SO(4). These are real and antisymmetric matrices. It is easily verified that $\rho^r_{\alpha\beta}$ is anti-selfdual, while $\eta^{r'}_{\alpha\beta}$ is selfdual. Their further properties are

$$\begin{aligned}
&\rho^r_{\alpha\gamma}\,(\rho^s)^\gamma{}_\beta = -\delta^{rs}\delta_{\alpha\beta} + \epsilon^{rst}\,\rho^t_{\alpha\beta}\, , && \text{idem } \eta^{r'}_{\alpha\beta}\, ,\\
&\rho^r_{\alpha\beta}\rho^r_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma} - \epsilon_{\alpha\beta\gamma\delta}\, , &&\\
&\eta^{r'}_{\alpha\beta}\eta^{r'}_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma} + \epsilon_{\alpha\beta\gamma\delta}\, , &&\\
&\epsilon^{trs}(\rho^r)_{\alpha\beta}\,(\rho^s)_{\gamma\delta} = \delta_{\beta\gamma}\,(\rho^t)_{\alpha\delta} + 3\text{ more}\, , && \text{idem } \eta^{r'}_{\alpha\beta}\ .
\end{aligned} \tag{A.4}$$

For SU(2) triplets, we use the notation:

$$X^{AB} = X^r\, T^r_{AB}\, , \qquad X^r = \tfrac{1}{2} X^{AB}\, T^r_{AB}. \tag{A.5}$$

B. The gauged Maurer-Cartan form and the C-functions

A convenient choice for the $\mathrm{Sp}(n_H,1)/\,\mathrm{Sp}(n_H)\times\mathrm{Sp}(1)$ coset representative L is [41]

$$L = \gamma^{-1}\begin{pmatrix} 1 & t^\dagger \\ t & \Lambda \end{pmatrix} \tag{B.1}$$

where t is an n_H-component quaternionic vector t^p $(p = 1, \ldots, n_H)$, and

$$\gamma = (1 - t^\dagger\, t)^{1/2}\, , \qquad \Lambda = \gamma\,(I - t\, t^\dagger)^{-1/2}\ . \tag{B.2}$$

Here, I is an $n_H \times n_H$ unit matrix, and $\dagger$ refers to quaternionic conjugation, and it can be verified that $\Lambda t = t$. The gauged Maurer-Cartan form is defined as

$$L^{-1} D_\mu L = \begin{pmatrix} Q_\mu & P^\dagger_\mu \\ P_\mu & Q'_\mu \end{pmatrix}\, , \tag{B.3}$$

where $D_\mu L$ is given in (2.7), with T^r representing three anti-hermitian quaternions (in the matrix representation of quaternions $T^r = -i\,\sigma^r/2$) obeying

$$[T^r, T^s] = \epsilon^{rst} T^t \tag{B.4}$$

and $T^{I'}$ represents a subset of $n_H \times n_H$ quaternion valued anti-hermitian matrices spanning the algebra of the subgroup $K \subset \mathrm{Sp}(n_H)$ that is being gauged. A direct computation gives

$$Q_\mu = \frac{1}{2}\gamma^{-2}\left(D_\mu t^\dagger t - t^\dagger D_\mu t\right) - A^r_\mu T^r \tag{B.5}$$

$$Q'_\mu = \gamma^{-2}\left(-tD_\mu t^\dagger + \Lambda D_\mu \Lambda + \tfrac{1}{2}\partial_\mu(t^\dagger t)I\right) - A^{I'}_\mu T^{I'}, \tag{B.6}$$

$$P_\mu = \gamma^{-2}\Lambda D_\mu t\,, \tag{B.7}$$

where

$$D_\mu t = \partial_\mu t + t\,T^r A^r_\mu - A^{I'}_\mu T^{I'}\,t\;. \tag{B.8}$$

The C functions are easily computed to yield

$$C^r = L^{-1}T^r L = \gamma^{-2}\begin{pmatrix} T^r & T^r t^\dagger \\ -tT^r & -tT^r t^\dagger \end{pmatrix} \tag{B.9}$$

$$C^{I'} = L^{-1}T^{I'}L = \gamma^{-2}\begin{pmatrix} -t^\dagger T^{I'} t & -t^\dagger T^{I'}\Lambda \\ \Lambda T^{I'} t & \Lambda T^{I'}\Lambda \end{pmatrix} \tag{B.10}$$

C. The model for $\mathrm{Sp}(1,1)/\,\mathrm{Sp}(1)\times\mathrm{Sp}(1)_R$

This coset, which is equivalent to $\mathrm{SO}(4,1)/\,\mathrm{SO}(4)$, represents a 4-hyperboloid H_4. In this case we have a single quaternion $t = \phi^\alpha\,\sigma_\alpha$, and the vielbein becomes

$$V_\alpha^{A'A} = \gamma^{-2}\,\sigma_\alpha^{A'A}\;. \tag{C.1}$$

It follows from the definitions (2.4) and (2.5) that

$$g_{\alpha\beta} = \frac{2}{(1-\phi^2)^2}\,\delta_{\alpha\beta}\,, \qquad J^r_{\alpha\beta} = \frac{2\,\rho^r_{\alpha\beta}}{(1-\phi^2)^2}\;. \tag{C.2}$$

We also introduce a basis in the tangent space of H_4

$$V_\alpha{}^{\underline{\alpha}} = \frac{\sqrt{2}}{1-\phi^2}\,\delta_\alpha^{\underline{\alpha}}\;. \tag{C.3}$$

The $\mathrm{Sp}(1)_R$ connection Q^r_μ can be found from (B.5) as

$$Q^r_\mu = -2\,\mathrm{tr}\,(Q_\mu T^r) = \frac{1}{1-\phi^2}\left(2\rho^r_{\alpha\beta}\partial_\mu\phi^\alpha\,\phi^\beta - A^r_\mu\right)\;. \tag{C.4}$$

With the above results at hand, the Lagrangian can be written as

$$\begin{aligned} e^{-1}\mathcal{L} = R &- \tfrac{1}{4}(\partial\varphi)^2 - \tfrac{1}{2}e^{\varphi}\,G_{\mu\nu\rho}G^{\mu\nu\rho} - \tfrac{1}{4}\,e^{\frac{1}{2}\varphi}\,F^r_{\mu\nu}\,F^{r\mu\nu} - \tfrac{1}{4}\,e^{\frac{1}{2}\varphi}\,F^{r'}_{\mu\nu}\,F^{r'\mu\nu} \\ &- \frac{4}{(1-\phi^2)^2}\,D_\mu\phi^\alpha D^\mu\phi^\beta\,\delta_{\alpha\beta} - \frac{6e^{-\frac{1}{2}\varphi}}{(1-\phi^2)^2}\,\left[g_R^2 + g'^2(\phi^2)^2\right]\,, \end{aligned} \tag{C.5}$$

where the covariant derivatives are defined as

$$D_\mu\phi^\alpha = \partial_\mu\phi^\alpha - \tfrac{1}{2}g_R A_\mu^r(\rho^r)^\alpha{}_\beta\,\phi^\beta - \tfrac{1}{2}g' A_\mu^{r'}(\eta^{r'})^\alpha{}_\beta\,\phi^\beta, \tag{C.6}$$

and we have re-introduced the gauge coupling constants g_R and g'. The supersymmetry transformation rules are

$$\delta\psi_\mu = D_\mu\varepsilon + \tfrac{1}{48}e^{\frac{1}{2}\varphi}G^+_{\nu\sigma\rho}\,\Gamma^{\nu\sigma\rho}\,\Gamma_\mu\,\varepsilon\,, \tag{C.7}$$

$$\delta\chi = \tfrac{1}{4}\left(\Gamma^\mu\partial_\mu\varphi - \tfrac{1}{6}e^{\frac{1}{2}\varphi}G^-_{\mu\nu\rho}\,\Gamma^{\mu\nu\rho}\right)\varepsilon\,, \tag{C.8}$$

$$\delta\lambda^r_A = -\tfrac{1}{8}F^r_{\mu\nu}\Gamma^{\mu\nu}\varepsilon_A - g_R\frac{e^{-\frac{1}{2}\varphi}}{1-\phi^2}\,T^r_{AB}\;\varepsilon^B\,, \tag{C.9}$$

$$\delta\lambda^{r'}_A = -\tfrac{1}{8}F^{r'}_{\mu\nu}\Gamma^{\mu\nu}\varepsilon_A + g' e^{-\frac{1}{2}\varphi}\frac{\phi^\alpha\phi^\beta}{1-\phi^2}\,(\bar\sigma_\alpha T^{r'}\sigma_\beta)_{AB}\;\varepsilon^B\,, \tag{C.10}$$

$$\delta\psi^{A'} = \frac{1}{1-\phi^2}\,D_\mu\phi^\alpha\,\sigma_a^{A'A}\,\varepsilon_A\,, \tag{C.11}$$

where $D_\mu\varepsilon_A = \nabla_\mu\varepsilon_A + Q^r_\mu(T^r)_A{}^B\varepsilon_B$, with ∇_μ containing the standard torsion-free Lorentz connection only, and Q^r is defined in (C.4).

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